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Transition behaviours in two coupled Josephson junction equations

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Abstract

The dynamics of two coupled Josephson junction equations are investigated via mathematical reasoning and numerical simulations. We show that for a fixed coupling K, the whole parameter space can be comparted into three regions: a quenching region, a synchronized running periodic region and a region where these two states coexist. It is further shown that with the increase of the coupling K, the system may transit from a synchronizing state to a quenching state. The characteristic of the critical line $K^*(b)$ which separates these two states is mathematically analysed.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Synchronization, quenching, and especially the coexistence of these two states, are themselves interesting nonlinear dynamical phenomena occurring in many systems, ranging from superconductor systems such as row switching and discrete breathers in Josephson junction arrays [3–6] to biological systems such as genetic toggle switches [1, 2], etc. It is important to clarify these states in parameter space and to understand the transition behaviours thoroughly. Unfortunately, there are no general methods which are valid for these complex systems.

As one of the classical physical systems, the *Frenkel–Kontorova* model which is composed of discrete coupled Josephson junction arrays is used to describe the underlying physical concepts of various systems [7]. Knowledge of its abundant transition behaviours will help us to understand the behaviours of complex systems more clearly. In [8], the *Frenkel–Kontorova* model is studied numerically, where the pinning–depinning transition is pursued as a function of the number of chain particles for different values of the interaction strengths.

Generally speaking, the collective dynamical behaviours of coupled systems mainly depend on two factors: one is the dynamics of single oscillators which compose the oscillatory media; the other is the interaction among them in the presence of mutual coupling. For a single oscillator, the dynamics can be well described by the motion of a single pendulum,

$$\ddot{x} + \alpha \dot{x} + \sin x = b, \tag{1}$$

where α is the damping coefficient, $b > 0$ is a constant driving force. This equation is also used to model the phase difference of a single Josephson junction with dc biased current. Numerical results of the dynamic of equation (1) have been treated in [12] and the references therein. For mathematical analysis of the dynamics, one can refer to [13]. Recently, a more clear figure of the transition behaviours of such a single system has been presented in [9], where the parameter space (b, α) was comparted into three parts, according to different dynamical behaviours. For a chain of N coupled oscillators, the transition behaviours will become much more complex. Nevertheless, we may take a step-by-step approach to first consider a model composed of two identical pendulums with diffusive coupling. The corresponding equations can be written as

$$\begin{cases} \ddot{x}_1 + \alpha \dot{x}_1 + \sin x_1 = K(x_2 - x_1) + b, \\ \ddot{x}_2 + \alpha \dot{x}_2 + \sin x_2 = K(x_1 - x_2), \end{cases} \tag{2}$$

where K is the coupling coefficient. For these two coupled oscillators, we try to understand their dynamical behaviours thoroughly, which may give insight into the transition behaviours in large systems.

For system (2) with enough large coupling and damping coefficients, it has been mathematically proved that the system behaves exactly like a one-dimensional system [10], which means that the two oscillators behave like a single one. However, the interesting dynamical behaviours will not only occur in such a parameter regime. In the present work, we will show that for $b > 2$ the two oscillators are synchronized, while for $1 < b < 2$ increasing the coupling strength K may lead the two oscillators from synchronizing states to quenching states. This is a different transition behaviour from the Kuramotor oscillators with globally coupling, where the system transits from quenching states to synchronizing states with the increase of K [11].

The paper is organized as follows. In section 2, we study different states of system (2) and the transition behaviours in (b, α) parameter space with fixed coupling strength. Section 3 is devoted to investigate the transition behaviours with the changing of the coupling strength. In section 4, we discuss future work related to system (2).

2. Dynamical states and transition behaviours for fixed coupling

In this section, we investigate the dynamical behaviours of system (2) for a fixed coupling coefficient. Equivalently, equations (2) can be rewritten in the following form:

$$\begin{cases} \dot{x}_1 = y_1, \\ \dot{y}_1 = -\alpha y_1 + b - \sin x_1 + K(x_2 - x_1), \\ \dot{x}_2 = y_2, \\ \dot{y}_2 = -\alpha y_2 - \sin x_2 + K(x_1 - x_2). \end{cases} \tag{3}$$

In [10], two kinds of periodicity of solutions are defined: one is a periodic solution and the other is a running periodic solution. To highlight our point about the synchronization, we list the definitions of them.

Definition 2.1. *The solution* $\{(x_1(t), y_1(t), x_2(t), y_2(t))\}_{t\geqslant 0}$ *to equations (3) with the initial condition* $x(0) = x_0$, $y(0) = y_0$ *is called a periodic solution in* R^4*, if there exists a smallest positive value* T*, s.t.*

$$x_i(t+T) = x_i(t), \qquad y_i(t+T) = y_i(t), \qquad i = 1, 2, \tag{4}$$

for any $t \geqslant 0$.

Definition 2.2. *A solution* $\{(x_1(t), y_1(t), x_2(t), y_2(t))\}_{t\geqslant 0}$ *to equations (3) with the initial condition* $x(0) = x_0$, $y(0) = y_0$ *is called a running periodic solution in* R^4*, if there exists a smallest positive value* T*, s.t.*

$$x_i(t+T) = x_i(t) + 2\pi, \qquad y_i(t+T) = y_i(t), \qquad i = 1, 2, \tag{4$'$}$$

for any $t \geqslant 0$.

Since we just concern the synchronization of the two oscillators, we only need to consider the phase difference of these two oscillators. We have the following theorem.

Theorem 2.1. *If* $K > 0$*, then for a solution* $\{(x_1(t), x_2(t))\}_{t\geqslant 0}$ *to equations (2) with the initial condition* $x_1(0) = x_1^0, x_2(0) = x_2^0$*, there exists a constant* $M > 0$*, such that for all time* $t \geqslant 0$*,*

$$|x_1(t) - x_2(t)| \leqslant M.$$

Proof. Let $u = x_1 - x_2$, and subtract the second equation from the first one in equations (2), we have

$$\ddot{u} + \alpha\dot{u} + 2Ku = b - (\sin x_1 - \sin x_2). \tag{5}$$

Equivalently, equation (5) can be written as

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -2Ku - \alpha v + f(t), \end{cases} \tag{5$'$}$$

where $f(t) = b - \sin x_1(t) + \sin x_2(t)$.

Then the solution to equation (5′) can be written as

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \mathrm{e}^{At} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \int_{-\infty}^{t} \mathrm{e}^{A(t-s)} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} \mathrm{d}s, \tag{6}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -2K & -\alpha. \end{pmatrix}.$$

Since A is stable, i.e., the real parts of the two eigenvalues of A are negative $\left(\lambda_{1,2} = \frac{-\alpha\pm\sqrt{\alpha^2-8K}}{2}\right)$, then there exists a constant $C > 0$, $0 < \sigma < -\max\{\mathrm{Re}\,\lambda_1, \mathrm{Re}\,\lambda_2\}$, such that

$$\|\mathrm{e}^{At}\| \leqslant C\,\mathrm{e}^{-\sigma t}\ (t \geqslant 0). \tag{7}$$

From equation (6), we know that there exist constants C_1 and C_2, such that

$$|u| \leqslant (|u| + |v|) \leqslant C_1\,\mathrm{e}^{-\sigma t} + C_2 \int_{-\infty}^{t} \mathrm{e}^{-\sigma(t-s)}\,\mathrm{d}s. \tag{8}$$

Hence there exists a constant $M > 0$ such that for any $t \geqslant 0$,

$$|x_1(t) - x_2(t)| \leqslant M. \qquad \square$$

It follows from theorem 2.1 that $\lim_{t\to\infty} \frac{x_1(t)-x_2(t)}{t} = 0$. This means that the two oscillators of system (2) are always frequency locked.

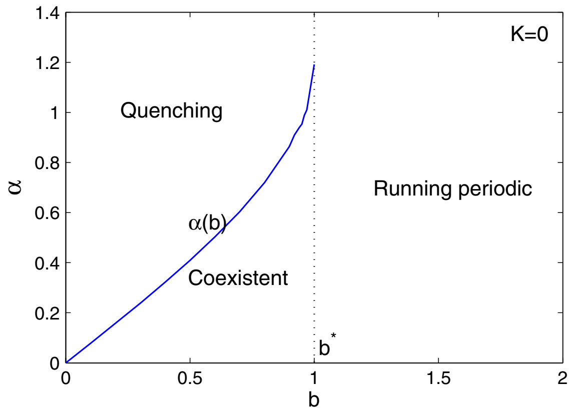

Figure 1. The transition line $b = 1$ and $\alpha = \alpha(b)$ which separate the space (b, α) into three parts. Here $K = 0$.

Furthermore, we have

Theorem 2.2. *System (2) has no periodic solutions.*

Proof. Multiplying $\dot{x_1}$ on the first equation of system (2) and $\dot{x_2}$ on the second one, and plus the two equations, we have

$$\ddot{x_1}\dot{x_1} + \ddot{x_2}\dot{x_2} - b \cdot \dot{x_1} + \dot{x_1} \cdot \sin x_1 + \dot{x_2} \cdot \sin x_2 + K(x_1 - x_2)(\dot{x_1} - \dot{x_2}) = -\alpha\left(\dot{x_1}^2 + \dot{x_2}^2\right). \quad (9)$$

Suppose that system (2) has a periodic solution with period T; then for any time $t_0 \geqslant 0$, we have

$$\int_{t_0}^{t_0+T} d\left[\frac{1}{2} \cdot \left(\dot{x_1}^2 + \dot{x_2}^2\right) - bx_1 - \cos x_1 - \cos x_2 + \frac{1}{2}K(x_1 - x_2)^2\right] = \int_{t_0}^{t_0+T} -\alpha\left(\dot{x_1}^2 + \dot{x_2}^2\right) \mathrm{d}t. \quad (10)$$

Because of the periodicity of $x(t)$, the left-hand side of equation (10) is zero. Then the right-hand side is zero too. Hence for $\alpha > 0$, $\dot{x_1}^2 + \dot{x_2}^2 = 0$. As a result, $y_1 = \dot{x_1} = 0$, $y_2 = \dot{x_2} = 0$. This contradicts the assumption. Therefore, system (2) has no periodic solution. □

Now fixing the coupling strength K, let us study the transition behaviours with the variations of damping coefficient α and the driving force b. For comparison, we first let $K = 0$, which means that there is only one oscillator. According to [9], the whole parameter space can be separated into three regions: one is for quenching state, another is for running periodic state and the third one is for the coexistence of these two states (see figure 1). There are two curves marked out for the transitions among three states: the straight line $b = 1$ which separates the quenching and the running behaviours, and the curve $\alpha = \alpha(b)$ below which quenching and running states coexist.

For the coupled case $K > 0$, numerical simulations show that there are no other solutions other than quenching and running solutions, and the transition behaviours with the variation of α and b are similar to the single case, except that the critical value of $b^*(K)$ is now not exactly equal to 1. We will prove in the following section that $b^*(K) > 1$, and $b^*(K)$ monotonically increases to value 2 as the coupling coefficient K increases to sufficiently large values.

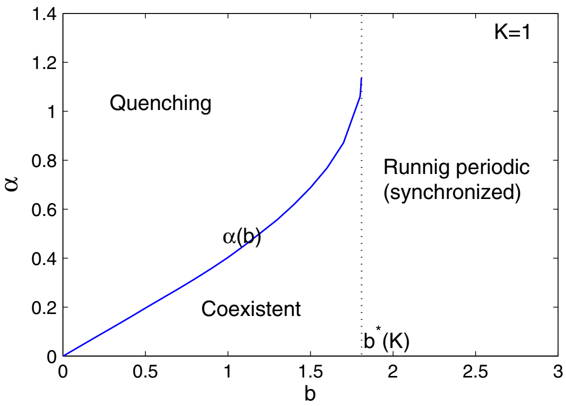

Figure 2. The transition line $b = b^*(K)$ and $\alpha = \alpha(b)$ which separate the space (b, α) into three parts. Here $K > 0$.

3. Transition behaviours with the increase of the coupling strength

In this section, we investigate the transition behaviours of system (3) with the variation of the coupling strength K.

3.1. Transition behaviours for $K > 1/\pi$

Firstly, we confine our discussion for $K > \frac{1}{\pi}$.

The equilibrium point to system (3) should satisfy the following conditions:

$$y_1 = y_2 = 0, \qquad F(x_1, x_2) = x_1 - \left(x_2 + \frac{1}{K}\sin x_2\right) = 0, \tag{11}$$

$$G(x_1, x_2) = x_2 - \left(x_1 + \frac{1}{K}\sin x_1\right) + \frac{b}{K} = 0. \tag{12}$$

Obviously, for $b > 2$, there is no equilibrium point for system (3), so we only need to consider the case $b \leqslant 2$. Equations (11) and (12) decide two smooth curves

$$L_1 : x_1 = x_2 + \frac{\sin x_2}{K} \tag{13}$$

and

$$L_2 : x_2 = x_1 + \frac{\sin x_1}{K} - \frac{b}{K} \tag{14}$$

in x_1–x_2 plane, respectively.

For the curve L_1, it is below the line $x_1 = x_2$ in the range $\{0 \leqslant x_2 \leqslant \pi, 0 \leqslant x_1 \leqslant 2\pi\}$ and is above the line $x_1 = x_2$ in the range $\{\pi \leqslant x_2 \leqslant 2\pi, 0 \leqslant x_1 \leqslant 2\pi\}$, while for the curve L_2, it is above the line $x_2 = x_1 - \frac{b}{K}$ in the range $\{0 \leqslant x_1 \leqslant \pi, 0 \leqslant x_2 \leqslant 2\pi\}$ and is below the line $x_2 = x_1 - \frac{b}{K}$ in the range $\{\pi \leqslant x_1 \leqslant 2\pi, 0 \leqslant x_2 \leqslant 2\pi\}$ (see figure 3). The situation is the same every 2π period of x_1 and x_2.

For a given coupling strength K, increasing the parameter b from zero to enough large will result in these two curves from intersecting with each other to leaving apart. Thus we have the following theorem.

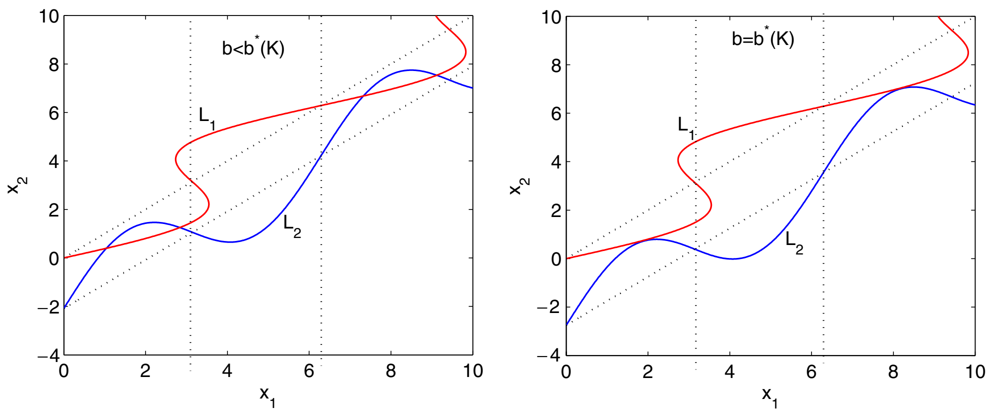

Figure 3. The relative positions of the curves L_1 and L_2 for $b < b^*(K)$ (left panel) and $b = b^*(K)$ (right panel).

Theorem 3.1. *For any fixed* $K > 0$*, there exists a unique value of the driving force* $b = b^*(K)$*, such that, for* $b > b^*(K)$*, system*(3) *has no equilibrium point.*

Furthermore, we have

Theorem 3.2. *The critical value* $b^*(K)$ *exhibits the following characteristics:*

(i) $b^*(K) > 1$*;*
(ii) For $K > \frac{1}{\pi}$*,* $b^*(K)$ *increases monotonically with the increase of K;*
(iii) $b^*(K) \to 2$ *as* $K \to \infty$*.*

Proof. In addition to (11) and (12), we have

$$L_3 : \sin x_1 + \sin x_2 = b. \tag{15}$$

Equation (15) determines a closed curve which is symmetric about the lines $x_1 = \pi/2$ and $x_2 = \pi/2$. The curves L_1 and L_2 are tangent with each other if and only if L_1 and L_3 are tangent at the same point.

(i) To prove $b^*(K) > 1$, we only need to show that, for $0 < b \leqslant 1$, there exists at least one fixed point for system (3). We will prove that, for $0 < b \leqslant 1$, the curves L_1 and L_3 intersect with each other in the region $0 \leqslant x_1 \leqslant \pi/2, 0 \leqslant x_2 \leqslant \pi/2$. In fact, in this region, the equations $F(x_1, x_2) = 0$ and $b - \sin x_1 + \sin x_2 = 0$ determine two explicit functions $x_2 = u_1(x_1, K)$ and $x_2 = u_2(x_1, K, b) = \arcsin(b - \sin x_1)$, respectively. The function $u_2(x_1, K, b) - u_1(x_1, K)$ is continuous with respect to x_1 in $[0, \pi/2]$, and has different signs at $x_1 = 0$ and $x_1 = \pi/2$, $u_2(x_1, K) \in [\arcsin(b - 1), \arcsin b)$ and it is monotonically decreasing with respect to x_1. Thus L_3 must intersect with L_1. Hence $b^*(K) > 1$.

(ii) From theorem 3.1, one knows that for any fixed value of K, there exists a critical value $b^*(K)$, such that the curves L_1 and L_3 are tangent. Suppose that the tangent point is $(x_1^*(K), x_2^*(K))$. With the assumption $K > \frac{1}{\pi}$, we know that the curve L_1 is in the range: $|x_1 - x_2| \leqslant \pi$, and for $1 < b < 2$, the curve L_3 is restricted in the region $2K\pi \leqslant x_1 \leqslant (2K+1)\pi, 2K\pi \leqslant x_2 \leqslant (2K+1)\pi$. Hence we can confine our discussion in the region $0 \leqslant x_1 \leqslant \pi, 0 \leqslant x_2 \leqslant \pi$. It is impossible for L_1 and L_3 to be tangent with each other except in the region $\pi/2 < x_1 \leqslant \pi, 0 \leqslant x_2 \leqslant \pi/2$. Confining in this region,

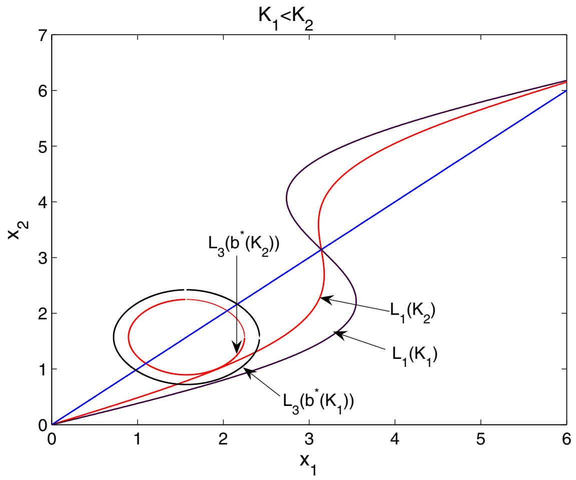

Figure 4. The profiles of the curves $L_1(K_1)$, $L_1(K_2)$, $L_3(b^*(K_1))$ and $L_3(b^*(K_2))$ in the range $\{0 \leqslant x_1 \leqslant 2\pi, 0 \leqslant x_2 \leqslant 2\pi\}$ for $\frac{1}{\pi} < K_1 < K_2$.

one knows that for any $\frac{1}{\pi} < K_1 < K_2$, $L_1(K_1)$ lies below the curve $L_1(K_2)$, then the curve $L_3(b^*(K_1))$ that is tangent with the curve $L_1(K_1)$ lies below the curve $L_3(b^*(K_2))$ that is tangent with the curve $L_1(K_2)$. To be more clear, see figure 4.

In the region $\pi/2 \leqslant x_1 \leqslant \pi, 0 \leqslant x_2 \leqslant \pi/2$, L_3 is a monotonic curve. And for $1 < b_1 < b_2$, $L_3(b_1)$ lies below the curve $L_3(b_2)$. Actually, L_3 is a closed curve which becomes contractive with the increase of b. Hence $b^*(K_1) < b^*(K_2)$.

(iii) When $K \to \infty$, the coordinates of tangent point satisfy

$$x_1^* - x_2^* = \frac{1}{K} \sin x_2^* \to 0.$$

Associating with the restrictive condition $\frac{\pi}{2} \leqslant x_1^* \leqslant \pi, 0 \leqslant x_2^* \leqslant \frac{\pi}{2}$, we can get $x_1^* \to \frac{\pi}{2}, x_2^* \to \frac{\pi}{2}$, which means that $b^*(K) = \sin x_1^* + \sin x_2^* \to 2$. □

According to theorem 2.2, the monotonic function $b = b^*(K)$ decides uniquely a reverse function $K = K^*(b)$ which is also monotonically increasing with the increase of b. Figure 5 shows the transition line of $K = K^*(b)$ in parameter space (K, b). This curve separates the whole space into two parts. For $K < K^*(b)$, the two coupled oscillators undergo synchronized running periodic motion whatever their initial conditions be (see figure 6(*a*)), $K > K^*(b)$, the system will become quench for the over-damped case (see figure 6(*b*)) and become coexistence of quench and running motion for the under-damped case.

The phenomenon that increasing the coupling strength K will lead the system transition from synchronizing to quenching state is contrary to the transition behaviour like Kuramoto transition [11] where increasing the coupling favours the realization of synchronization.

3.2. Transition behaviours for $K < 1/\pi$

It should be pointed out that when the coupling K is weak, the number of equilibrium points will increase. Then we can not mathematically prove that the critical curve $b = b^*(K)$ is monotonic. In the following, we will numerically discuss the transition behaviours for $K < 1/\pi$.

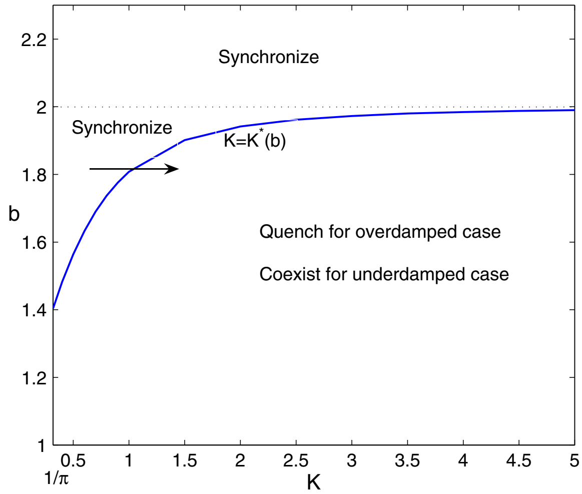

Figure 5. For $K > 1/\pi$, the transition curve $K = K^*(b)$ separates the whole space (K, b) into two parts.

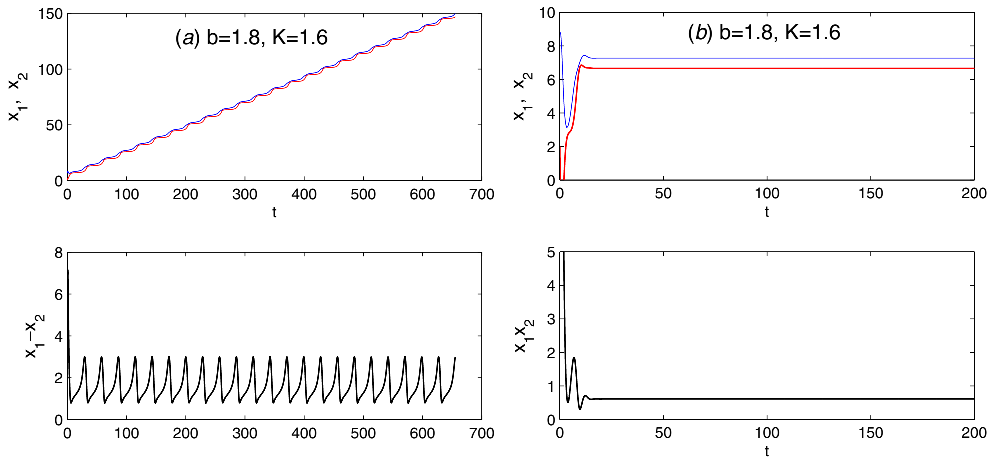

Figure 6. The trajectories of $\{x_1(t)\}$ and $\{x_2(t)\}$ and their phase difference for (a) $K < K^*(b)$, (b) $K > K^*(b)$ of system (2) for $\alpha = 2$.

In figure 7, we plot the critical line $b = b^*(K)$, above which the two oscillators are synchronized and below which the system quenches. One can see that, for weak coupling, $b^*(K)$ is not a monotonic function of K. From another point of view, we are informed from figure 7 that for a fixed value of b (about $1.3 < b < 1.97$), increasing the weak coupling strength K properly will lead the two oscillators transit from synchronized running motion to a quenching state and again to a synchronized running motion. This is an intriguing phenomenon.

4. Discussion

In this paper, we have studied the dynamical behaviours of two diffusively coupled oscillators. For $b \geqslant 2$, the two oscillators will always undergo synchronized running periodic motions.

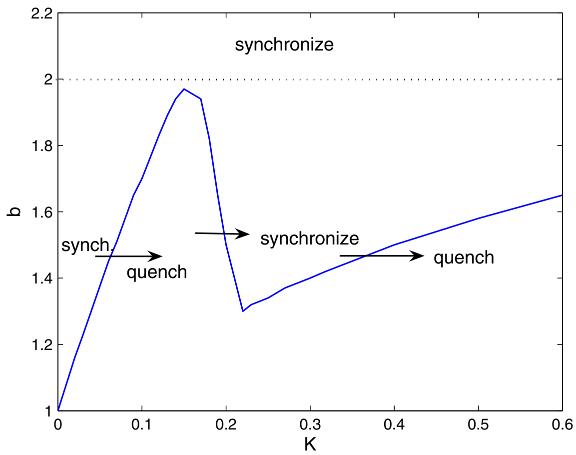

Figure 7. Sketch of the transition curve $b = b^*(K)$ for $0 < K \leqslant 1/\pi$.

However, for $1 < b < 2$, the coupled system displays interesting transition behaviours. Especially, in the case $K > 1/\pi$, the system will transit from synchronizing state to quenching state with the increase of the coupling strength. The transition line $K = K^*(b)$ was proved to be a monotonically increasing curve with the increase of driving force b.

If we further introduce a periodic driving, then based on the results manifested in figure 5, one can expect many complex behaviours both for $K \leqslant K^*(b)$ and $K > K^*(b)$. It is interesting to investigate whether chaotic synchronization will occur for the case $K < K^*(b)$.

One can further investigate the phenomenon of stochastic resonance (SR) in the presence of noise. For the single case, we have shown in our previous paper [9] that in a regime where the system is quenching (see the quenching regime in figure 1), interwell SR or intrawell SR will occur depending on the type of the equilibrium point. Then for the coupled case with a fixed coupling strength, we anticipate a better effect of SR in the same parameter regime (see the quenching regime in figure 2) than in the single case. Figure 5 also informs us to observe array-enhanced SR in the regime $K > K^*(b)$. One can further study the variation of the SR effect with the increase of the coupling in this parameter regime.

What we have found here may be extended to N diffusively coupled systems. We guess that N coupled systems will also produce quenching and synchronizing states, and the transitions between these states are essentially similar to the two coupled oscillators.

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